On the superintegrable Richelot systems.

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Abstract

We introduce the Richelot class of superintegrable systems in N-dimensions whose $n \leq N$ equations of motion coincide with the Abel equations on $n-1$ genus hyperellipic curve. The corresponding additional integrals of motion are the second order polynomials of momenta and multiseparability of the Richelot superintegrable systems is related with classical theory of covers of the hyperelliptic curves.

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In the antique rarely-read collections of scientific societies as well as in comprehensive scientific correspondence of the scientist of the past an enormous quantity of scientific matter is contained, from which anyone capable can find something motivating to start their own work, as well as simultaneously learn something useful.

K. Weierstrass, "The speech delivered upon assuming the position of Rector of Berlin University on October 15, 1873", Phys. Usp. 42 1219 (1999)

1 Introduction

In classical mechanics, superintegrable systems are characterized by the fact that they possess more than N integrals of motion functionally independent, globally defined in a $2N$-dimensional phase space. In particular, when the number of integrals is $2N-1$, the systems are said to be maximally superintegrable. The dynamics of these systems is particularly interesting: all bounded orbits are closed and periodic [5]. The phase space topology is also very rich: it has the structure of a symplectic bifoliation, consisting of the usual Liouville-Arnold invariant fibration by Lagrangian tori and of a (coisotropic) polar foliation [23].

The notion of superintegrability possesses an interesting analog in quantum mechanics. Sommerfeld and Bohr were the first to notice that systems allowing separation of variables in more than one coordinate system may admit additional integrals of motion. Superintegrable systems show accidental degeneracy of the energy levels, which can be removed by taking into account the quantum numbers associated to the additional integrals of motion, some of their bound state energy levels may be calculated algebraically and the corresponding wave functions are expressed in terms of polynomials. One of the best examples of this phenomenon is provided by the harmonic oscillator and the Kepler-Coulomb problem. A large number of papers have been published on super-integrability in these last years, most of them related with second-order integrals of motion (see [3, 8, 10, 14, 17, 20, 26, 30, 31] for some recent results and an extensive list of references).

A systematic investigations of superintegrable systems have a very long story, which began in 1761 when Euler proposed construction of the additional algebraic integral for the differential equation

$$\frac{d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}} \pm \frac{d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}} = 0,$$

where f is an arbitrary quartic [11]. The corresponding superintegrable Stäckel systems have been classified in [17].

The Abel theorem may be regarded as a generalization of these Euler results. Remind that the Abel equations

$$\sum_{j=1}^{n} \frac{u_i(\mathrm{x}_j)\, d\mathrm{x}_j}{\sqrt{f(\mathrm{x}_j)}} = 0, \qquad i = 1, \ldots, p, \tag{1.1}$$

play a pivotal role in classical mechanics and that there are two approaches to investigation of the Abel equations associated with the Jacobi and Richelot, respectively (see thirtieth lecture in the Jacobi book [12]). In modern mathematics, the first approach or the Abel-Jacobi map is one of the main constructions of algebraic geometry which relates an algebraic curve to its Jacobian variety. The second approach yields addition theorems theory, moduli theory (modular equations), cryptography and so on.

The aim of this note is to discuss the Richelot construction of addition integrals for the Abel equations and construction of the corresponding N-dimensional superintegrable systems in classical mechanics. We treat only classical superintegrable systems here, though the corresponding results for the quantum systems follow easily.

The paper is organized as follows. In Section II, the main Richelot results are briefly reviewed. Then we discuss possible application of these results to classification of the superintegrable Stäckel systems. In Section III, the classification of superintegrable systems separable in orthogonal coordinate systems is treated and solved. Some open problems are discussed in the final Section.

2 The Richelot superintegrable systems

In this section we use the original Richelot notations [25].

Let y be the algebraic function of x defined by an equation of the form

$$\Phi(\mathrm{x}, \mathrm{y}) = \mathrm{y}^m + f_1(\mathrm{x})\mathrm{y}^{m-1} + \cdots + f_m(\mathrm{x}) = 0, \tag{2.1}$$

where $f_1(\mathrm{x}), \ldots, f_m(\mathrm{x})$ are rational polynomials in x. According to the Abel theorem a system of the p differential equations

$$\frac{du_i}{d\mathrm{x}_1}\, d\mathrm{x}_1 + \cdots + \frac{du_i}{d\mathrm{x}_N} d\mathrm{x}_N = 0, \qquad i = 1, \ldots, p$$

have additional algebraic integrals if $N > p$ and if $u_1, \cdots, u_p$ being a set of linearly independent abelian integrals of the first kind on algebraic curve (2.1).

The problem of the determining of these integrals consists only in the expression of the fact that $\mathrm{x}_1, \ldots, \mathrm{x}_N$ constitute a set belonging to a lot of coresidual sets of places, so we have some determinant representations for additional integrals.

For the particular forms of the curve (2.1) there are some explicit formulae due by Euler [11], Lagrange [22], Jacobi [13], Richelot [25], Weierstrass [32] and some other [2, 7, 15].

2.1 The Richelot integrals

Following to Richelot [25] we will consider hyperelliptic curve

$$\mathrm{y}^2 = f(\mathrm{x}) \equiv A_{2n}\mathrm{x}^{2n} + A_{2n-1}\mathrm{x}^{2n-1} + \cdots + A_1\mathrm{x} + A_0 \tag{2.2}$$

and the following system of $n-1$ differential equations

$$\begin{aligned}
&\frac{d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}} + \frac{d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}} + \cdots + \frac{d\mathrm{x}_n}{\sqrt{f(\mathrm{x}_n)}} = 0,\\
&\frac{\mathrm{x}_1 d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}} + \frac{\mathrm{x}_2 d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}} + \cdots + \frac{\mathrm{x}_n d\mathrm{x}_n}{\sqrt{f(\mathrm{x}_n)}} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{\mathrm{x}_1^{n-2} d\mathrm{x}_1}{\sqrt{f(\mathrm{x}_1)}} + \frac{\mathrm{x}_2^{n-2} d\mathrm{x}_2}{\sqrt{f(\mathrm{x}_2)}} + \cdots + \frac{\mathrm{x}_n^{n-2} d\mathrm{x}_n}{\sqrt{f(\mathrm{x}_n)}} = 0\,.
\end{aligned} \tag{2.3}$$

Let a_k be the values of x at the branch points of the curve (2.2) and $F(\mathrm{x}) = (\mathrm{x}-\mathrm{x}_1)(\mathrm{x}-\mathrm{x}_2)\cdots(\mathrm{x}-\mathrm{x}_n)$, then in generic case additional integrals of the Abel equations (2.3) are equal to

$$C_k = \frac{\left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}\cdot\frac{1}{a_k-\mathrm{x}_1}+\cdots+\frac{\sqrt{f(\mathrm{x}_n)}}{F'(\mathrm{x}_n)}\cdot\frac{1}{a_k-\mathrm{x}_n}\right]^2}{\left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}+\cdots+\frac{\sqrt{f(\mathrm{x}_n)}}{F'(\mathrm{x}_n)}\right]^2 - A_{2n}}\,F(a_k) \tag{2.4}$$

If $A_{2n} = 0$ additional integrals of equations (2.3) look like

$$C_k = \left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}\cdot\frac{1}{a_k-\mathrm{x}_1}+\cdots+\frac{\sqrt{f(\mathrm{x}_n)}}{F'(\mathrm{x}_n)}\cdot\frac{1}{a_k-\mathrm{x}_n}\right]^2\sqrt{F(a_k)}\,. \tag{2.5}$$

There are $n-1$ functionally independent integrals of motion C_k and, of course, their combinations are integrals of motion too.

Using special combinations of C_k we can avoid calculations of the values a_k of x at the branch points [13, 25, 32]. As an example, in his paper Richelot found the following two algebraic integrals

$$K_1 = \left[\frac{\sqrt{f(\mathrm{x}_1)}}{F'(\mathrm{x}_1)}+\cdots+\frac{\sqrt{f(\mathrm{x}_n)}}{F'(\mathrm{x}_n)}\right]^2 - A_{2n-1}(\mathrm{x}_1+\cdots+\mathrm{x}_n) - A_{2n}(\mathrm{x}_1+\cdots+\mathrm{x}_n)^2 \tag{2.6}$$

and

$$K_2 = \left[\frac{\sqrt{f(\mathrm{x}_1)}}{\mathrm{x}_1^2F'(\mathrm{x}_1)}+\cdots+\frac{\sqrt{f(\mathrm{x}_n)}}{\mathrm{x}_n^2F'(\mathrm{x}_n)}\right]^2\mathrm{x}_1^2\mathrm{x}_2^2\cdots\mathrm{x}_n^2 - A_1\left(\frac{1}{\mathrm{x}_1}+\cdots+\frac{1}{\mathrm{x}_n}\right) - A_0\left(\frac{1}{\mathrm{x}_1}+\cdots+\frac{1}{\mathrm{x}_n}\right)^2. \tag{2.7}$$

The generating function of additional integrals was proposed by Weierstrass [32], see [2] for detail.

2.2 Construction of the Richelot superintegrable systems

Let as apply the Richelot construction to classification of the superintegrable systems in classical mechanics.

Definition 1 *The N-dimensional integrable system is the superintegrable Richelot system if $n-1$, $1 < n \le N$, equations of motion are the Abel-Richelot equations (2.3).*

It's easy to get a lot of such superintegrable Richelot systems in framework of the Jacobi separation of variables method, see [17, 30, 31].

Let us start with the maximally superintegrable Richelot systems at $N = n$. In this case construction consists of the one hyperelliptic curve (2.2)

$$\mu^2 = f(\lambda), \qquad \text{where} \qquad f(\lambda) = A_{2n}\lambda^{2n} + A_{2n-1}\lambda_i^{2n-1} + \cdots + A_1\lambda + A_0, \tag{2.8}$$

and n arbitrary substitutions

$$\lambda_j = v_j(q_j) \qquad \mu_j = u_j(q_j)p_j, \qquad j = 1,\ldots,n, \tag{2.9}$$

where p and q are canonical variables $\{p_j, q_i\} = \delta_{ij}$.

The n copies of this hyperelliptic curve and these substitutions give us n separated relations

$$p_j^2\,u_j^2(q_j) = A_{2n}v_j(q_j)^{2n} + A_{2n-1}v_j(q_j)_i^{2n-1} + \cdots + A_1v_j(q_j) + A_0, \qquad j = 1,\ldots,n, \tag{2.10}$$

where $2n+1$ coefficients $A_{2n},\ldots,A_0$ are linear functions of n integrals of motion $H_1,\ldots,H_n$ and $2n+1$ parameters $\alpha_0,\ldots,\alpha_{2n+1}$.

Solving these separated equations with respect to H_k one gets functionally independent integrals of motion

$$H_k = \sum_{j=1}^{n} (S^{-1})_{jk} \Big(p_j^2 + U_j(q_j) \Big) , \qquad k = 1, \ldots, n = N, \tag{2.11}$$

where $U_j(q_j)$ are so-called Stäckel potentials and S is the Stäckel matrix [27].

If H_1 is the Hamilton function, then coordinates $q_j(t, \alpha_1, \ldots, \alpha_n)$ are determined from the Jacobi equations

$$\sum_{j=1}^{n} \int \frac{S_{1j}(q_j) dq_j}{\sqrt{\sum_{k=1}^{n} \alpha_k S_{1j}(q_j) - U_j(q_j)}} = \tau - t \,, \tag{2.12}$$

and

$$\sum_{j=1}^{n} \int \frac{S_{ij}(q_j) dq_j}{\sqrt{\sum_{k=1}^{n} \alpha_k S_{kj}(q_j) - U_j(q_j)}} = \beta_i \,, \qquad i = 2, \ldots, n \,, \tag{2.13}$$

where t is the time variable conjugated to the Hamilton function H_1. According to Jacobi [12] these equations are another form of the Abel equations (1.1) and describe inversion of the corresponding Abel map.

In order to use the Richelot results we have to impose some constraints on the entries of the Stäckel matrix $S_{kj}(q_j)$, which give rise to some restrictions on the coefficients A_k [17, 30].

Namely, if we compare $n-1$ equations (2.3) and equations (2.13) at $\lambda = \mathrm{x}$ one gets that the Stäkel matrix in λ variables has to be one of the following matrices

$$S^{(k)} = \begin{pmatrix} \lambda_1^k & \lambda_2^k & \cdots & \lambda_n^k \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \qquad k = n, n+1, \ldots, 2n, \tag{2.14}$$

so that

$$\mu^2 = f(\lambda) = \lambda^k H_1 + \lambda^{n-1} H_{n-1} \cdots + H_{n-1}\lambda + H_n + \sum_{j=0}^{2n} \alpha_j \lambda^j \,. \tag{2.15}$$

Such as k is arbitrary number from n to $2n$ we have a family of the dual Stäckel systems associated with one hyperelliptic curve (2.8) and different blocks of the corresponding Brill-Noether matrix [28, 29]

Remark 1 For any two dual systems with Hamiltonians H_1 and $\widetilde{H}_1$ the corresponding Stäckel matrices $S^{(k)}$ and $S^{(\widetilde{k})}$ are distinguished on the first row only. These Stäckel systems are related by canonical transformation of the time $t \to \widetilde{t}$:

$$\widetilde{H}_1 = \mathrm{v}(q)\, H_1, \quad d\widetilde{t} = \mathrm{v}(q)\, dt, \qquad \text{where} \qquad \mathrm{v}(q) = \frac{\det S^{(k)}}{\det S^{(\widetilde{k})}}. \tag{2.16}$$

Such dual systems have common trajectories with different parametrization by the time [29, 20]. Existence of the such dual systems is related with the fact that the Abel map is surjective and generically injective.

Remark 2 For the dual systems the corresponding hyperelliptic curves (2.15) are related by permutation of one of the α's and Hamiltonian H_1 and, therefore, such transformations are called the coupling constant metamorphoses [6, 18, 29]. Such transformations are related with the reciprocal transformations as well [1].

Remark 3 There exist the Richelot superintegrable systems that can be solved via separation of variables in more than one coordinate system. These systems are associated with non-isomorphic curves whose Jacobians are isomorphic to one another (either Jacobian of (2.2) could be isomorphic to a strata of another Jacobian or Jacobian of (2.2) could be isogenous to a product of some different curves etc). Such curves have already occurred in the work of Hermite, Goursat, Burkhardt, Brioschi, and Bolza, see Krazer [21] and a lot of modern works on the Frey-Kani covers.

Now let as briefly consider construction of the superintegrable Richelot systems for which $n-1$ equations of motion among the N equations of motion are the Abel-Richelot equations only. In this case to n separated relations (2.10) have to be complimented by $N-n$ separated relations

$$\Phi_m(p_m, q_m, H_1, \ldots, H_N) = 0, \qquad n < m \le N.$$

Solving this complete set of the separated equations with respect to integrals of motion H_k we have to get N functionally independent integrals of motion (2.11). As above the Abel equations have to coincide with the Richelot equations (2.3) and, therefore, the $n \times n$ block of the $N \times N$ Stäckel matrix has to be matrix as (2.14). If we take into account all these restrictions one gets complete classifications of the superintegrable Stäckel-Richelot systems.

The main problem is that we want to get Hamiltonians H_j in some physical variables x instead of Hamiltonians (2.11) in terms of the abstract separated variables q. According to [17, 30, 31] it leads to some additional restrictions on the coefficients A_j in (2.8) and substitutions (2.9).

It easy to see that the Stäckel integrals of motion H_k (2.11) and the Richelot additional integrals of motion are the second order polynomials in momenta

$$K_1 = \left[\frac{u_1 p_1}{F'(v_1)} + \cdots + \frac{u_n p_n}{F'(v_n)}\right]^2 - A_{2n-1}(v_1 + \cdots + v_n) - A_{2n}(v_1 + \cdots + v_n)^2 \tag{2.17}$$

and

$$K_2 = \left[\frac{u_1 p_1}{v_1^2 F'(v_1)} + \cdots + \frac{u_n p_n}{v_n^2 F'(v_n)}\right]^2 v_1^2 v_2^2 \cdots v_n^2 - A_1\left(\frac{1}{v_1} + \cdots + \frac{1}{v_n}\right) - A_0\left(\frac{1}{v_1} + \cdots + \frac{1}{v_n}\right)^2 . \tag{2.18}$$

Here u_j and v_j are functions on coordinates only.

So, in the Stäckel-Richelot case all the integrals of motion are the second order polynomials in momenta and it allows us to find natural Hamiltonian superintegrable systems on the Riemannian manifolds using well-studied theory of the orthogonal coordinate systems and the corresponding Killing tensors [4, 9, 19, 24].

3 The Richelot systems separable in orthogonal coordinate systems

All the orthogonal separable coordinate systems can be viewed as an orthogonal sum of certain basic coordinate systems [4, 9, 19, 24]. Below we consider some of these basic coordinate systems in the n-dimensional Euclidean space only.

3.1 The basic orthogonal coordinate systems

Definition 2 *The elliptic coordinate system $\{q_i\}$ in the N-dimensional Euclidean space $\mathbb{E}_N$ with parameters $e_1 < e_2 < \cdots < e_N$ is defined through the equation*

$$e(\lambda) = 1 + \sum_{k=1}^{N} \frac{x_k^2}{\lambda - e_k} = \frac{\prod_{j=1}^{N}(\lambda - q_j)}{\prod_{i=1}^{N}(\lambda - e_i)} . \tag{3.19}$$

The defining equation (3.19) should be interpreted as an identity with respect to λ.

It is possible to degenerate the elliptic coordinate systems in a proper way by letting two or more of the parameters e_i coincide. Then the ellipsoid will become a spheroid, or even a sphere if all parameters coincide. Rotational symmetry of dimension m is thus introduced if $m+1$ parameters coincide.

Example 1 As an example when $e_1 = e_2$, we have

$$e(\lambda) = 1 + \frac{r^2}{\lambda - e_1} + \sum_{i=3}^{N} \frac{x_i^2}{\lambda - e_i} = \frac{\prod_{i=1}^{N-1}(\lambda - q_i)}{\prod_{j=1}^{N-1}(\lambda - e_j)}, \qquad r^2 = x_1^2 + x_2^2 . \tag{3.20}$$

It defines elliptic coordinate system in $\mathbb{E}_{N-1} = \{r, x_3, \cdots, x_N\}$. In order to get an orthogonal coordinate system $\{q_1, \cdots, q_N\}$ in $\mathbb{E}_N$, we could complement r with an angular coordinate q_N in the $\{x_1, x_2\}$-plane, for instance through

$$x_1 = r\cos q_N, \qquad x_2 = r\sin q_N\,, \qquad \text{where} \quad r = \sqrt{\operatorname{res}|_{\lambda=e_1}\, e(\lambda)}\,. \tag{3.21}$$

At $N = 3$ these equations define the prolate spherical coordinate system.

When $e_1 = e_2 = \cdots = e_n$ the only remaining coordinate is $r = \sqrt{\sum x_i^2}$ and $N-1$ angular coordinates have to be introduced on the unit sphere $\mathbb{S}_{N-1}$. According to [19] these angular coordinates are so-called ignorable coordinates.

Definition 3 *The parabolic coordinate system $\{q_i\}$ in $\mathbb{E}_N$ with parameters $e_1 < e_2 < \cdots < e_{N-1}$ is defined through the equation*

$$e(\lambda) = \lambda - 2x_N - \sum_{k=1}^{N-1} \frac{x_k^2}{\lambda - e_k} = \frac{\prod_{j=1}^{N}(\lambda - q_j)}{\prod_{i=1}^{N-1}(\lambda - e_i)}. \tag{3.22}$$

This orthogonal coordinate system can, in fact, be derived from the elliptic coordinate system as well. Namely, substitute

$$x_i = \frac{x_i'}{\sqrt{e_i}}, \qquad i = 1, \ldots, N-1, \qquad x_N = \frac{x_N' - e_N}{\sqrt{e_N}}$$

into the (3.19) and let e_N tend to infinity, then drop the primes one gets the parabolic coordinate system.

The parabolic coordinate system can be degenerated in the same way as the elliptic coordinate system.

Example 2 If $e_1 = e_2$, we have

$$e(\lambda) = \lambda - 2x_N - \frac{r^2}{\lambda - e_1} - \sum_{k=3}^{N-1} \frac{x_k^2}{\lambda - e_k} = \frac{\prod_{j=1}^{N-1}(\lambda - q_j)}{\prod_{i=1}^{N-2}(\lambda - e_i)}, \qquad r^2 = x_1^2 + x_2^2\,. \tag{3.23}$$

As above in order to get an orthogonal coordinate system $\{q_1, \cdots, q_n\}$ in $\mathbb{E}_N$, we could complement r with an angular or ignorable coordinate q_N in the $\{x_1, x_2\}$-plane defined by (3.21). At $N = 3$ it is so-called rotational parabolic coordinates.

Definition 4 *The elliptic coordinate system $\{q_i\}$ on the sphere $\mathbb{S}_N$ with parameters $e_1 < e_2 < \cdots < e_{N+1}$ is defined through the equation*

$$e(\lambda) = \sum_{k=1}^{N+1} \frac{x_k^2}{\lambda - e_k} = \frac{\prod_{j=1}^{N}(\lambda - q_j)}{\prod_{i=1}^{N+1}(\lambda - e_i)}\,. \tag{3.24}$$

Notice that (3.24) implies $\sum_{i=1}^{N+1} x_i^2 = 1$. In the similar manner we can define elliptic coordinate system $\{q_i\}$ on the hyperboloid $\mathbb{H}_N$ with $x_0^2 - \sum_{i=1}^{N} x_i^2 = 1$ [19]. As above these coordinates can be degenerated by letting some, but not all, parameters e_i coincide.

Remark 4 There are some algorithms [4, 24] and software [16] that for a given natural Hamilton function $H = T + V$ determine if separation coordinates exist, and in that case, show how to construct them, i.e. how to get determining function $e(\lambda)$.

3.2 The maximally superintegrable Richelot systems

The basic orthogonal coordinate systems is defined by the function

$$e(\lambda) = \frac{\prod_{i=1}^{N}(\lambda - q_j)}{\prod_{j=1}^{M}(\lambda - e_j)} = \frac{\phi(\lambda)}{u(\lambda)} \qquad M = N, N \pm 1, \tag{3.25}$$

which is the ratio of the following polynomials

$$\phi(\lambda) = \prod_{i=1}^{N}(\lambda - q_j), \qquad \text{and} \qquad u(\lambda) = \prod_{j=1}^{M}(\lambda - e_j)\,. \tag{3.26}$$

We can describe the maximally superintegrable Richelot systems separable in these coordinate systems using the following Proposition.

Proposition 1 *If $n = N$ separated relations have the following form*

$$p_i^2\, u(q_i)^2 = \frac{1}{2}\left[u(\lambda)\cdot\left(H_1\,\lambda^k + \sum_{i=2}^{N} H_i\,\lambda^{n-i}\right) - \alpha(\lambda)\right]_{\lambda=q_i}, \qquad \alpha(\lambda) = \sum_{j=0}^{2N} \alpha_j\,\lambda^j, \tag{3.27}$$

where $\alpha(\lambda)$ is arbitrary polynomial, then equations of motion (2.13) are the Abel-Richelot equations (2.3).

If $k = n$ the corresponding maximally superintegrable Hamiltonian H_1

$$H_1 = T + V = \sum_{i=1}^{N} \mathrm{res}\Big|_{\lambda=q_i} \frac{1}{e(\lambda)}\cdot p_i^2 - \sum_{i=1}^{N} \mathrm{res}\Big|_{\lambda=q_i} \frac{\alpha(\lambda)}{u^2(\lambda) e(\lambda)}$$

has a natural form in Cartesian coordinates in $\mathbb{E}_n$

$$H_1 = T + V = \frac{1}{2}\sum_{i=1}^{N} p_{x_i}^2 + \sum_{i=0}^{M} \mathrm{res}\Big|_{\lambda=e_i} \frac{\alpha(\lambda)}{u^2(\lambda)\, e(\lambda)}. \tag{3.28}$$

Here we introduce additional parameter $e_0 = \infty$.

If $k > n$ then $H_1^{(k>n)} = \mathrm{v}(x)\, H_1$, where function $\mathrm{v}(x)$ is defined by (2.16).

It is easy to prove, that these maximally superintegrable Richelot systems coincide with the well-known superintegrable systems [3, 8, 10, 14, 20, 26]. l For elliptic coordinate system in $\mathbb{E}_N$ equation (3.28) yields the following potential

$$V = \alpha_{2N}(x_1^2 + \cdots x_n^2) + \sum_{i=1}^{N} \frac{\gamma_i}{x_i^2}, \qquad \gamma_i = \frac{\alpha(e_i)}{\prod_{j\neq i}(e_i - e_j)^2}\,.$$

For parabolic coordinate system in $\mathbb{E}_N$ one gets

$$V = \alpha_{2N}(x_1^2 + \cdots 4x_N^2) + \gamma_N x_N + \sum_{i=1}^{N-1} \frac{\gamma_i}{x_i^2}, \qquad \gamma_N = 4\alpha_{2N} \sum e_i + 2\alpha_{2N-1}\,.$$

For elliptic coordinate system on the sphere $\mathbb{S}_N$ or on the hyperboloid $\mathbb{H}_N$ we obtain

$$V = \sum_{i=1}^{N+1} \frac{\gamma_i}{x_i^2}, \qquad \gamma_i = \frac{\alpha(e_i)}{\prod_{j\neq i}(e_i - e_j)^2}\,.$$

Example 3 Let us consider parabolic coordinates (q_1, q_2, q_3) defined by

$$e(\lambda) = \lambda - 2x_3 - \frac{x_1^2}{\lambda - e_1} - \frac{x_2^2}{\lambda - e_2} = \frac{(\lambda - q_1)(\lambda - q_2)(\lambda - q_3)}{(\lambda - e_1)(\lambda - e_2)}\,,$$

whereas the corresponding momenta are equal to

$$p_i = \frac{x_1 p_{x_1}}{2(q_i - e_1)} + \frac{x_2 p_{x_2}}{2(q_i - e_2)} + \frac{p_{x_3}}{2}, \qquad i = 1, \ldots, 3.$$

In this case the separated relations (3.27-3.33) look like

$$p_i^2 (q_i - e_1)^2 (q_i - e_2)^2 = \frac{1}{2}\Big[(H_1\lambda^2 + H_2\lambda + H_3)(\lambda - e_1)(\lambda - e_2) - \alpha(\lambda)\Big]_{\lambda = q_i}, \qquad i = 1, \ldots, 3. \quad (3.29)$$

Solving these equations with respect to H_k one gets integral of motion and the following Hamilton function

$$H_1 = \frac{p_{x_1} + p_{x_2} + p_{x_3}}{2} + \alpha_6(x_1^2 + x_2^2 + 4x_3^2) + \gamma_3 x_3 + \frac{\gamma_1}{x_1^2} + \frac{\gamma_2}{x_2^2} + const\,. \quad (3.30)$$

It is maximally superintegrable Hamiltonian with the Stäckel integrals of motion H_2, H_3 and two additional Richelot integral of motion $K_{1,2}$ (2.17-2.18):

$$\begin{aligned} K_1 &= \left(\frac{(q_1 - e_1)(q_1 - e_2)p_1}{(q_1 - q_2)(q_1 - q_3)} + \frac{(q_2 - e_1)(q_2 - e_2)p_2}{(q_2 - q_1)(q_2 - q_3)} + \frac{(q_3 - e_1)(q_3 - e_2)p_3}{(q_3 - q_1)(q_3 - q_2)} \right)^2 \\ &+ \frac{\alpha_5}{2}(q_1 + q_2 + q_3) + \frac{\alpha_6}{2}(q_1 + q_2 + q_3)^2\,, \\ K_2 &= \left(\frac{(q_1 - e_1)(q_1 - e_2)p_1}{(q_1 - q_2)(q_1 - q_3)q_1^2} + \frac{(q_2 - e_1)(q_2 - e_2)p_2}{(q_2 - q_1)(q_2 - q_3)q_2^2} + \frac{(q_3 - e_1)(q_3 - e_2)p_3}{(q_3 - q_1)(q_3 - q_2)q_3^2} \right)^2 q_1^2 q_2^2 q_3^2 \\ &+ \frac{H_3 e_1 + (H_3 - H_2 e_1)e_2}{2}\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right) - \frac{e_1 e_2 H_3}{2}\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right)^2 . \end{aligned} \quad (3.31)$$

In physical variables (x, p_x) these integrals have more complicated structure.

It is easy to prove that integrals H_1, H_2, H_3 and K_1, K_2 are functionally independent. of course, all these integrals of motion may be obtained in framework of the Weierstrass approach [32] as well.

Example 4 Now let us consider dual Stäckel system and put $k = n + 1$ in the Stäckel matrix (2.14) from the previous Example. It means that we change one of the coefficients in the separated relations (3.29) and consider the following separated relations

$$p_i^2 (q_i - e_1)^2 (q_i - e_2)^2 = \frac{1}{2}\Big[(\widetilde{H}_1\lambda^3 + \widetilde{H}_2\lambda + \widetilde{H}_3)(\lambda - e_1)(\lambda - e_2) - \alpha(\lambda)\Big]_{\lambda = q_i}, \qquad i = 1, \ldots, 3.$$

Solving these equations one gets superintegrable system with the Hamiltonian

$$\widetilde{H}_1 = \mathrm{v}(q)\, H_1 = \frac{1}{2x_3 + e_1 + e_2}\, H_1,$$

where H_1 is given by (3.30). Of course, this canonical transformation of time changes additional integrals of motion $K_{1,2}$ (3.30).

3.3 The superintegrable Richelot systems

Now let us consider degenerate coordinate systems for which two or more of the parameters e_j coincide.

In terms of the separated coordinates defining function $e(\lambda)$ remains meromorphic function with n simple roots and $m = n, n \pm 1$ simple poles. For the construction of the Richelot systems we need degenerations such that $1 < n < N$.

In this case in order to get superintegrable Richelot systems with $n-1$ additional integrals of motion we have to take n separated relations (3.27)

$$p_i^2\, u(q_i)^2 = \frac{1}{2}\left[u(\lambda)\cdot\left(H_1\,\lambda^k + \sum_{i=2}^{n} H_i\,\lambda^{n-i}\right) - \alpha(\lambda) + \frac{1}{2}\sum_{j=n+1}^{N}\frac{u(\lambda)}{\mathrm{g}_j(\lambda)}H_j\right]_{\lambda=q_i}, \qquad (3.32)$$

and $N-n$ separated relations for ignorable variables

$$p_j^2 = 2\Big(U_j(q_j) - H_j\Big), \qquad j = n+1,\ldots,N. \qquad (3.33)$$

Here polynomials $\mathrm{g}_j(\lambda)$ depend on degree of degeneracy and definition of the ignorable variables [4, 19], whereas $U_j(q_j)$ are arbitrary functions on these ignorable (angular) variables q_j.

Solving these equations with respect to integrals of motion H_j one gets the Hamilton function in the same form as (3.28) in which, roughly speaking, trailing coefficient of the polynomial $\alpha(\lambda)$ depends on ignorable variables.

Proposition 2 *For degenerate elliptic or parabolic coordinates superintegrable potentials have the following form (3.28)*

$$V = \sum_{i=0}^{m} \mathrm{res}\Big|_{\lambda=e_i} \frac{\alpha(\lambda) - U_i}{u^2(\lambda)\, e(\lambda)}, \qquad e_0 = \infty, \qquad (3.34)$$

where $U_i = 0$ for single roots e_i of initial function $(\lambda - e_1)\cdots(\lambda - e_M)$ (3.26) after degeneration $e_k = e_j$. For degenerate roots $e_k = e_j$ potential U_i are arbitrary functions on the corresponding ignorable variables.

It allows us classify all the superintegrable Richelot systems using known classification of the orthogonal coordinate systems [3, 8, 10, 14, 17, 20, 26].

Example 5 Let us consider prolate spherical coordinate system (q_1, q_2, q_3) defined by

$$e(\lambda) = 1 + \frac{x_1^2 + x_2^2}{\lambda - e_1} + \frac{x_3^2}{\lambda - e_3} = \frac{(\lambda - q_1)(\lambda - q_2)}{(\lambda - e_1)(\lambda - e_3)}, \qquad q_3 = \arctan\left(\frac{x_1}{x_2}\right).$$

The corresponding momenta are

$$p_1 = \frac{x_1 p_{x_1} + x_2 p_{x_2}}{2(q_1 - e_1)} + \frac{x_3 p_{x_3}}{2(q_1 - e_3)}, \qquad p_2 = \frac{x_1 p_{x_1} + x_2 p_{x_2}}{2(q_2 - e_1)} + \frac{x_3 p_{x_3}}{2(q_2 - e_3)}, \qquad p_3 = x_2 p_{x_1} - x_1 p_{x_2}\,.$$

In this case $\mathrm{g}(\lambda) = (e_3 - e_1)^{-1}(\lambda - e_1)$ and the separated relations (3.32-3.33) look like

$$p_i^2 (q_i - e_1)^2 (q_i - e_3)^2 = \frac{1}{2}\left[(H_1\lambda + H_2)(\lambda - e_1)(\lambda - e_2) - \alpha(\lambda) + \frac{(\lambda - e_3)(e_3 - e_1)H_3}{2}\right]_{\lambda=q_i},$$

$$p_3 = 2\,(\,U\,(q_3) - H_3\,)\,,$$

where $\alpha(\lambda) = \alpha_4\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$.

Solving these equations with respect to H_k one gets integrals of motion and the following Hamilton function

$$H_1 = \frac{p_{x_1} + p_{x_2} + p_{x_3}}{2} + \alpha_4(x_1^2 + x_2^2 + x_3^2) + \frac{\gamma_1 - U\left(\dfrac{x_1}{x_2}\right)}{x_1^2 + x_2^2} + \frac{\gamma_3}{x_3^2} - 2\alpha_4(e_3 + e_1) - \alpha_3\,,$$

where

$$\gamma_{1,3} = \frac{\alpha(e_{1,3})}{(e_1 - e_3)^2}\,.$$

It is superintegrable Hamiltonian with the Stäckel integrals of motion H_2, H_3 and additional Richelot integral of motion K_1 (2.17), which is equal to

$$K_1 = \left(\frac{(q_1-e_1)(q_1-e_3)p_1}{q_1-q_2} + \frac{(q_2-e_1)(q_2-e_3)p_2}{q_2-q_1}\right)^2 - \frac{(H_1-\alpha_3)(q_1+q_2)}{2} + \frac{\alpha_4(q_1+q_2)^2}{2}\,.$$

In physical variables (x, p_x) one gets the following expression for this integral of motion

$$K_1 = \frac{(x_1p_{x_1}+x_2p_{x_2}+x_3p_{x_3})^2}{4} + \frac{e_1+e_3-x_1^2-x_2^2-x_3^2}{2}\Big(\alpha_4(e_1+e_3-x_1^2-x_2^2-x_3^2)+\alpha_3-H_1\Big)\,.$$

The second Richelit integral K_2 (2.18) looks like

$$K_2 = \left(\frac{(q_1-e_1)(q_1-e_3)p_1}{(q_1-q_2)q_1^2} + \frac{(q_2-e_1)(q_2-e_3)p_2}{(q_2-q_1)q_2^2}\right)^2 q_1^2q_2^2 - A_1\left(\frac{1}{q_1}+\frac{1}{q_2}\right) - A_0\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\,,$$

where

$$A_1 = \frac{1}{2}\left(e_1e_3H_1-(e_1+e_3)H_2+(e_1-e_3)H_3-\alpha_1\right), \qquad A_0 = \frac{1}{2}\left(e_1e_3H_2-e_3(e_1-e_3)H_3-\alpha_0\right)\,.$$

Of course, substituting $H_1, \ldots, H_3$ into K_2 one gets that $K_1 = K_2$, because in this case we have only one Abel-Richelot equation, i.e. $n-1=1$. It means that Hamiltonian H_1 in $\mathbb{E}_3$ does not maximally superintegrable.

Example 6 Let us consider rotational parabolic coordinates (q_1, q_2, q_3) defined by

$$e(\lambda) = \lambda - 2x_3 - \frac{x_1^2+x_2^2}{\lambda-e_1} = \frac{(\lambda-q_1)(\lambda-q_2)}{\lambda-e_1}, \qquad q_3 = \arctan\left(\frac{x_1}{x_2}\right)\,,$$

whereas the corresponding momenta look like

$$p_1 = \frac{x_1p_{x_1}+x_2p_{x_2}}{2(q_1-e_1)} + \frac{p_{x_3}}{2}, \qquad p_2 = \frac{x_1p_{x_1}+x_2p_{x_2}}{2(q_2-e_1)} + \frac{p_{x_3}}{2}, \qquad p_3 = x_2p_{x_1}-x_1p_{x_2}\,.$$

In this case $\mathrm{g}(\lambda) = (\lambda-e_1)$ and the separated relations (3.32-3.33) are equal to

$$p_{1,2}^2(q_{1,2}-e_1)^2 = \frac{1}{2}\left[(H_1\lambda+H_2)(\lambda-e_1)-\alpha(\lambda)+\frac{H_3}{2}\right]_{\lambda=q_{1,2}}\,,$$

$$p_3^2 = 2(U(q_3)-H_3)\,,$$

where $\alpha(\lambda) = \alpha_4\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0$.

Solving these equations with respect to H_k one gets integrals of motion and the following Hamilton function

$$H_1 = \frac{p_{x_1}+p_{x_2}+p_{x_3}}{2} + \alpha_4(x_1^2+x_2^2+4x_3^2) + 2(2\alpha_4e_1+\alpha_3)x_3 + \frac{\alpha(e_1)-U\left(\dfrac{x_1}{x_2}\right)}{x_1^2+x_2^2} - 3\alpha_4e_1^2 - 2\alpha_3e_1 - \alpha_2\,.$$

It is superintegrable Hamiltonian with the Stäckel integrals of motion H_2, H_3 and additional Richelot integral of motion K_1 (2.17), which is equal to

$$\begin{aligned} K_1 &= \left(\frac{(q_1-e_1)p_1}{q_1-q_2} + \frac{(q_2-e_1)p_2}{q_2-q_1}\right)^2 + \frac{\alpha_3}{2}(q_1+q_2) + \frac{\alpha_4}{2}(q_1+q_2)^2 \\ &= \frac{p_{x_3}^2}{4} + 2\alpha_4x_3^2 + (2\alpha_4e_1+\alpha_3)x_3 + \frac{e_1(\alpha_4e_1+\alpha_3)}{2}\,. \end{aligned} \tag{3.35}$$

As above $K_1 = K_2$ (2.17-2.18) in this case.

Example 7 Let us consider degenerate elliptic coordinate system on the sphere $\mathbb{S}_3$ in $\mathbb{E}_4$, so that coordinates (q_1, q_2, q_3) are defined by

$$e(\lambda) = \frac{x_1^2 + x_2^2}{\lambda - e_1} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} = \frac{(\lambda - q_1)(\lambda - q_2)}{(\lambda - e_1)(\lambda - e_3)(\lambda - e_4)}, \qquad q_3 = \arctan\left(\frac{x_1}{x_2}\right).$$

It means that radius of the sphere is equal to $R = \sum_{i=1}^4 x_i^2 = 1$.

In this case $\mathrm{g}(\lambda) = (e_3 - e_1)^{-1}(e_1 - e_4)^{-1}(\lambda - e_1)$ and pair of the separated relations have the common form

$$\begin{aligned} p_i^2 (q_i - e_1)^2 (q_i - e_3)^2 (q_i - e_4)^2 &= \frac{1}{2}\Big[(H_1\lambda + H_2)(\lambda - e_1)(\lambda - e_3)(\lambda - e_4) - \alpha(\lambda) \\ &+ (e_3 - e_1)(e_1 - e_4)(\lambda - e_3)(\lambda - e_4) H_3\Big]_{\lambda = q_{1,2}}, \end{aligned} \tag{3.36}$$

where $\alpha(\lambda)$ is fourth order polynomial with arbitrary coefficients and third separated relation is equal to

$$p_3^2 = 2(U(q_3) - H_3).$$

Solving separated equations with respect to H_k one gets integrals of motion and the following Hamilton function

$$H_1 = \frac{1}{2}\left(\sum_{i=1}^4 x_i^2 \cdot \sum_{i=1}^4 p_i^2 - \left(\sum_{i=1}^4 x_i p_i\right)^2\right) + \frac{\gamma_1 + U\left(\frac{x_1}{x_2}\right)}{x_1^2 + x_2^2} + \frac{\gamma_3}{x_3^3} + \frac{\gamma_4}{x_4^2} - \frac{\alpha_4}{R}, \qquad \gamma_i = \frac{\alpha(e_i)}{\prod_{j \neq i}(e_i - e_j)^2}.$$

It is superintegrable Hamiltonian and additional Richelot integrals of motion looks like

$$\begin{aligned} K_1 &= \left(\frac{(q_1 - e_1)(q_1 - e_3)(q_1 - e_4)p_1}{q_1 - q_2} + \frac{(q_2 - e_1)(q_2 - e_3)(q_2 - e_4)p_2}{q_2 - q_1}\right)^2 \\ &+ \frac{(e_1 + e_3 + e_4)H_1 + \alpha_3 - H_2}{2}\Big(q_1 + q_2\Big) + \frac{\alpha_4 - H_1}{2}\Big(q_1 + q_2\Big)^2. \end{aligned} \tag{3.37}$$

In this case $n = 2$ and, therefore, $K_1 = K_2$ (2.17-2.18).

In this case change of the time (2.16) at $k = n + 1$ yields the following transformation of pair of the separated relation (3.32)- (3.36)

$$p_i^2\, u(q_i)^2 = \frac{1}{2}\left[u(\lambda)\cdot\big(H_1\,\lambda^2 + H_2\big) - \alpha(\lambda) + \frac{1}{2}\frac{u(\lambda)}{\mathrm{g}_3(\lambda)} H_3\right]_{\lambda = q_i} = \frac{H_1}{2}\,\lambda^5 + \ldots\Big|_{\lambda = q_i}.$$

In the right hand side of this equations we obtain $2n + 1$-order polynomial in λ and, therefore, the corresponding pair of the Abel equations are no longer the Richelot equations (2.3). This change of the time preserves integrability, but destroys superintegrability.

4 Conclusion

According to [17, 30, 31] there are two classes of superintegrable systems for which the angle variables are either *logarithmic* or *elliptic* functions. In the both cases one gets additional single-valued integrals of motion using addition theorems, which are particular cases of the Abel theorem.

The main aim of this note is to discuss one of the oldest but almost completely forgotten in modern literature Richelot's approach to construction and to investigation of the superintegrable systems separable in orthogonal coordinate systems. Of course, these n-dimensional superintegrable systems may be obtained using another known methods (see [3, 8, 10, 14, 20, 26] and references within). Nevertheless we think that new definition (3.28),(3.34)

$$V = \sum \operatorname{res}\Big|_{\lambda = e_i} \frac{\alpha(\lambda)}{u^2(\lambda)\, e(\lambda)}, \qquad u(\lambda) = \prod_{j=1}^M (\lambda - e_j),$$

of the superintegrable potentials through defining function $e(\lambda)$ of coordinate system and arbitrary polynomial $\alpha(\lambda)$ may be useful in applications.

It will be interesting to get quantum counterparts of the Richelot integrals of motion and to study the algebra of integrals of motion in the algebro-geometric terms. Another perspective consists in the classification of the Richelot superintegrable systems on the Darboux spaces.

One more important issue concerns relation of multiseparability of the Richelot superintegrable systems with classical theory of covers of the hyperelliptic curves.

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